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DYNAMICS OF A HOLLOW SYMMETRICALLY LOADED CIRCULAR ELASTIC CYLINDER

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A solution is presented for the dynamic axisymmetric problem of elasticity theory for a hollow elastic circular cylinder which permits determination of both the displacements and the stresses at any point of the cylinder during symmetric loading of its side surface.

The problem of the natural vibrations of a hollow elastic thickwalled cylinder has been considered in [1, 2]. A transcendental equation has been derived in [1] to obtain the natural frequencies in the case of the axisymmetric vibrations of the cylinder, and the propagation of elastic waves in a hollow circular cylinder has been investigated in [2] on the basis of the general elasticity equations.

A numerical method for solving problems of such a class has been elucidated in [3].

Proceeding from the Lamé equations, it is easy to show that an investigation of the state of stress-strain of a hollow cylinder under the assumption that the forcing terms are distributed over the volume, will reduce to the solution of the following problem (in dimensionless variables):

$$\frac{\partial \mathbf{u}_n(r,t)}{\partial r} = \mathrm{A}_n(r)\,\mathbf{u}_n(r,t) + \mathrm{B}\mathbf{G}_n(r,t) + \mathrm{B}\frac{\partial^2 \mathbf{u}_n(r,t)}{\partial t^2} \tag{1}$$

$$\mathrm{N}\mathbf{u}_n = 0 \quad \text{for } r = 1,\ r = \alpha \tag{2}$$

$$\mathbf{u}_n = 0, \quad \frac{\partial u_{zn}}{\partial t} = \frac{\partial u_{rn}}{\partial t} = 0 \quad \text{for } t = 0 \tag{3}$$

$$B = \begin{Vmatrix} 0 & 0 \\ E_0 & 0 \end{Vmatrix}, \quad \mathrm{N} = \begin{Vmatrix} 0 & 0 \\ 0 & E_0 \end{Vmatrix}, \quad \mathbf{G}_n = \begin{Vmatrix} G_{zn} \\ G_{rn} \\ 0 \\ 0 \end{Vmatrix}, \quad \mathrm{A}_n(r) = \| a_{ij}(r) \|$$

$$\mathbf{u}_n = \{u_{zn}/r_1,\ u_{rn}/r_1,\ \tau_n/E,\ \sigma_{rn}/E\}$$

$$a_{12} = -a_{43} = n,\ a_{13} = 2(1+\nu),\ a_{21} = -a_{34} = -\frac{n\nu}{1-\nu},\ a_{22} = -\frac{\nu}{(1-\nu)r}$$

$$a_{24} = \frac{(1+\nu)(1-2\nu)}{1-\nu},\quad a_{31} = \frac{n^2}{1-\nu^2},\quad a_{32} = a_{41} = \frac{\nu n}{(1-\nu^2)r},\quad a_{33} = -\frac{1}{r}$$

$$a_{42} = \frac{1}{(1-\nu^2)r^2},\quad a_{44} = -\frac{1-2\nu}{(1-\nu)r},\quad a_{11} = a_{14} = a_{23} = a_{43} = 0$$

The linear dimensions are here referred to the inner radius of the cylinder r_1, the time to $\sqrt{\rho / E}\, r_1$, where α is the ratio between the outer and inner radii, n is the wave number, E, ρ, ν are the elastic modulus, material density and Poisson's ratio, respectively, $G_{zn}(r,t)$, $G_{rn}(r,t)$ are body forces [4, 5], referred to E / r_1^2, and E_0 is a unit 2×2 matrix.

We hence have for the longitudinal and radial displacements

$$u_{zn}(z, r, t) = u_{zn}(r, t) \sin nz, \quad u_{rn}(z, r, t) = u_{rn}(r, t) \cos nz$$

and for the shear and normal stresses

$$\tau_n(z, r, t) = \tau_n(r, t) \sin nz, \quad \sigma_n(z, r, t) = \sigma_n(r, t) \cos nz$$

Superposition of these solutions permits consideration of the dynamic problem in the case of a symmetric loading of the form $g(z) f(t)$ (the subscript n is henceforth omitted).

Applying a generalized complex Fourrier transform to (1)

$$u_v^*(r, \omega) = \int_0^\infty u(r, t) e^{-vt} e^{i\omega t} dt \tag{4}$$

$$u(r, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u^*(r, \omega + iv) e^{-i\omega t} e^{vt} d\omega$$

and using condition (3), we convert the system (1) into

$$\partial \mathbf{u}^* / \partial r = [A(r) - \omega^2 B] \mathbf{u}^* + B\mathbf{G}^*(r, \omega) \tag{5}$$

with the boundary conditions

$$N\mathbf{u}^* = 0 \quad \text{for } r = 1, r = \alpha \tag{6}$$

Let us proceed from the following problem

$$d\mathbf{u}/dr = A(r)\mathbf{u} + \mathbf{g}(r) \tag{7}$$

$$N\mathbf{u} = 0 \quad \text{for } r = 1, r = \alpha \tag{8}$$

Here $\mathbf{g}(r)$ is a four-dimensional column vector whose components are continuous functions in $[1, \alpha]$.

We take the solution of (7) in the form [6]

$$\mathbf{u}(r) = c_1\mathbf{u}_1(r) + c_2\mathbf{u}_2(r) + \int_1^\alpha W(r, x)\,\mathbf{g}(x)\,dx, \quad W = \| w_{ij}(r, x) \| \tag{9}$$

$$w_{ij} = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{k=1}^{2} u_{ik}(r) v_{kj}(x) + \sum_{k=3}^{4} u_{ik}(r) v_{kj}(x), & 1 \leqslant x < r \leqslant \alpha \\ -\dfrac{1}{2} \displaystyle\sum_{k=1}^{2} u_{ik}(r) v_{kj}(x), & 1 \leqslant r < x \leqslant \alpha \end{cases}$$

Here $\| u_{ik}(r) \| = U(r)$ is the fundamental matrix of the homogeneous system corresponding to (7), $\| v_{ik}(r) \| = U^{-1}(r)$ is the inverse matrix.

It is easy to verify that if the first two solutions of the fundamental system satisfy conditions (8) at the left end, then (9) will also satisfy (8) for $r = 1$. For such a fundamental system, we obtain a system of algebraic equations to determine the arbitrary constants c_1, and c_2 from the boundary conditions on the right end. We have

$$c_m = -\frac{1}{2} \int_1^\alpha \sum_{i=3}^{4} v_{mi}(x) g_i(x)\,dx - \frac{1}{\Delta(\alpha)} \sum_{i=3}^{4} \sum_{l\,3}^{4} \sum_{s=3}^{4} \Delta_{lm}(\alpha) \times \tag{10}$$

$$u_{ls}(\alpha)\int_1^\alpha v_{si}(x)\,g_i(x)\,dx \quad \cdot (m=1,\,2)$$

$$\Delta(\alpha)=\begin{vmatrix} u_{31}(\alpha) & u_{32}(\alpha) \\ u_{41}(\alpha) & u_{42}(\alpha) \end{vmatrix}$$

Here $\Delta_{ij}(\alpha)$ are cofactors of the elements $u_{ij}(\alpha)$. Substituting (10) into (9), we obtain

$$\mathbf{u}(r)=\int_1^\alpha [P(r,\,x)+Q(r,\,x)]\,\mathbf{g}(x)\,dx \tag{11}$$

$$p_{ij}(r,\,x)=\begin{cases} \sum_{k=3}^{4} u_{ik}(r)\,v_{kj}(x), & 1\leqslant x<r\leqslant\alpha \\ -\sum_{k=1}^{2} u_{ik}(r)\,v_{kj}(x), & 1\leqslant r<x\leqslant\alpha \end{cases} \tag{12}$$

$$q_{ij}(r,\,x)=-\frac{1}{\Delta(\alpha)}\sum_{s=1}^{2}\sum_{p=3}^{4}\sum_{l=3}^{4}\Delta_{ls}(\alpha)\,u_{lp}(\alpha)\,u_{is}(r)\,v_{pj}(x) \tag{13}$$

As follows from (9), a nontrivial solution of the homogeneous system corresponding to (5), (6) is possible only for definite values of the parameter $\omega=\omega_k$, for which

$$\Delta(\alpha)=\Delta(\alpha,\,\omega_k)=0 \tag{14}$$

The solution of (14) is carried out numerically. The values of ω_k form an infinite sequence of eigenfrequencies, where there are no multiple frequencies. For a fixed ω_k a fundamental matrix $\|u_{ij}(r)\|$ can be constructed for the homogeneous system of differential equations corresponding to the problem dependent on the parameter ω as follows:

$$d\mathbf{u}/dr=(A(r)-\omega^2 B)\mathbf{u}+\mathbf{q}(r) \tag{15}$$

Under the boundary conditions (8), the solution of the system (15) can be represented on the basis of (11) and the expansion of the meromorphic function (13) as

$$\mathbf{u}(r,\,\omega)=\int_1^\alpha [P(r,\,x,\,\omega^2)+\varphi(\omega)\,g(x)]\,dx+\sum_{k=1}^{\infty}\int_1^\alpha \frac{R(r,\,x,\,\omega_k^2)}{\omega_k^2-\omega^2}\,\mathbf{g}(x)\,dx \tag{16}$$

$$R_{ij}(r,\,x,\,\omega_k^2)=-\operatorname{Res}\omega_k^2\,[q_{ij}(r,\,x,\,\omega^2)]=\frac{1}{\Delta'(\alpha,\,\omega_k^2)}\times$$

$$\sum_{s=1}^{2}\sum_{p=3}^{4}\sum_{p=3}^{4}\Delta_{ls}(\alpha,\,\omega_k^2)\,u_{lp}(\alpha,\,\omega_k^2)\,u_{is}(r,\,\omega_k^2)\,v_{pj}(x,\,\omega_k^2)$$

Here $P(r,\,x,\,\omega)$ has the form (12), and $\varphi(\omega)$ is an entire function. Using the second formula of (4) for the inverse transform of the solution (16), we have by the Jordan lemma

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-i\omega t}\,d\omega}{\omega_k^2-(\omega+iv)^2}=e^{-vt}\,\frac{\sin\omega_k t}{\omega_k}$$

and therefore, the solution of the initial problem (1)–(3) becomes [7]

$$\mathbf{u}(r,\,t)=\sum_{k=1}^{\infty}\int_1^\alpha \frac{R(r,\,x,\,\omega_k^2)}{\omega_k}\int_0^t \sin\omega_k(t-\tau)\,BG(x,\,\tau)\,dx\,d\tau \tag{17}$$

It should be noted that (17), (4) can be used to determine the displacements and stresses of an elastic cylinder under a symmetric dynamic loading of its side surface since it is easy to pass to boundary conditions of the type (2) and the inhomogeneous initial differential equations from the inhomogeneous boundary conditions in this case.

Table 1

n	$\alpha = 1.25$	$\alpha = 1.5$	$\alpha = 2.0$	$\alpha = 2.5$
1	0.8194	0.7637	0.6798	0.6324
2	1.1416	1.1137	1.0859	1.0679
3	7.8925	4.0867	2.2914	1.7550
4	14.4995	7.1678	3.5140	2.3288
5	15.7460	8.0599	4.2839	3.0546

The first five lowest dimensionless natural vibration frequencies $\lambda = r_1\omega\sqrt{\rho / E}$ calculated in conformity with the algorithm elucidated are presented in Table 1 for $\nu = 0.3$ and $n = 1$. A comparison of the frequencies presented resulted in complete agreement with the results of computations based on the transcendental equation obtained analogously to [1].

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